GENERAL METHOD OF INTEGRAL RELATIONS AND ITS APPLICATION TO BOUNDARY LAYER THEORY

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1. INTRODUCTION

IN the solution of equations containing partial derivatives, there has recently been widespread development of approximation methods for converting partial differential equations into systems of ordinary differential equations. These methods give greater accuracy compared with the method of finite differences and permit the use of very well developed numerical methods of solution of ordinary differential equations.

At the same time these methods are extremely suitable for application to electronic calculating machines: they are usually "stored" as standard sub-routines and so do not load the memory of the machine, as do for example all the various methods which use successive approximations or iterations.

Of the methods for reducing partial differential equations to systems of ordinary differential equations, the best developed are the "direct" method and the method of "integral relations". In the direct method the partial derivative with respect to one of the variables (for simplicity, we will consider partial differential equations with two independent variables only) is substituted by a finite difference relation which is based on certain interpolation formulae, expressing the value of the function at any point, in terms of the value of the function at the boundaries of the strips into which the interval is divided.

In the method of integral relations the initial differential equation containing partial derivatives is integrated first across the strips, whence the partial derivatives with respect to one variable are eliminated. After this, it is not the derivatives but the integrals of unknown functions which are represented with the help of interpolation formulae.

As experience has shown in the solution of numerous concrete problems in mechanics and physics, the method of integral relations permits the attainment of completely satisfactory accuracy even when the whole region, within which the solution is sought, is only divided into a very small number of strips. Often this accuracy is quite astonishing. For example, in a series of problems in gas dynamics, the division of the interval into two strips gives a solution correct to 1 per cent.

It is difficult to give a strict theoretical explanation of this unusually high accuracy, since the interpolation formulae both for integrals and for derivatives are capable of a similar order of accuracy, according to the width of the strip. The fact is that in practical calculations, use is never made of a very large number of strips, for which case the error would approach zero. On the other hand, the mathematician, in practice, always chooses that method which would give satisfactory accuracy for a comparatively wide strip. For a small number of interpolation points the accuracy of the approximate representation of the integral can be essentially higher than the approximate representation of the derivative. It is obvious that it is not difficult to reduce the primary functions, when the approximate representation of the integral is more exact than the approximate representation of the derivative, and vice versa; but it seems that the physically real quantities possess a known degree of continuity, whilst the continuity of the derivatives of these quantities does not depend on their physical character. From this it follows that the integral of a physical quantity is represented, with the help of interpolation formulae, considerably more accurately than its derivative, for a small number of interpolation points.

Below we will see that the method of "direct" and the method of "integral" relations differ in their degree of "smoothing" of the function. In the smoothing process, details of the behaviour of the function become less essential, and the greater the degree of smoothing, the more roughly can the function be represented, in order to derive from it its "smoothed" value with satisfactory accuracy.

2. GENERAL INTEGRAL RELATIONS

We will look at a system of differential equations with partial derivatives of the following type

$$\frac{\partial}{\partial x} P_i(x, y; u_1, u_2, \ldots, u_n) + \frac{\partial}{\partial y} Q_i(x, y; u_1, u_2, \ldots, u_n) = F_i(x, y; u_1, \ldots, u_n) \tag{2.1}$$

$$i = 1, 2, \ldots, n$$

where $u, u_2, \ldots, n_n$ are unknown functions, P_i, Q_i, F_i are given functions of their arguments.

It is necessary to find the solution of this system within the limits $a \leqslant x \leqslant b$; $c \leqslant y \leqslant d$. ("$a$" can tend to $-\infty$, "b" to $+\infty$). Let us multiply equation (2.1) by a certain function $f(y)$ and integrate it with respect to "y" over the interval (from "c" to "d"). We will obtain:

$$\left.\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\int_c^d f(y)\,P_i\,\mathrm{d}y+f(d)\,Q_i[x,d;u_1(x,d),\ldots,u_n(x,d)]-\\&-f(c)\,Q_i[x,c;u_1(x,c),\ldots,u_n(x,c)]-\int_c^d f'(y)\,Q_i\,\mathrm{d}y=\\&\qquad=\int_c^d f(y)\,F_i\,\mathrm{d}y\end{aligned}\right\}\tag{2.2}$$

Equation (2.2) is the starting relation in the "general method of integral relations".

This relation can easily be generalized for the case where the boundary of the interval is not linear. For example, let the upper limit be curvilinear:

$$y=\delta(x)\tag{2.3}$$

then in place of the relationship (2.2), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\int_c^\delta f(y)\,P_i\,\mathrm{d}y-\delta'\cdot f(\delta)\,P_i[x,\delta;u_1(x,\delta),\ldots,u_n(x,\delta)]+\\&+f(\delta)\,Q_i[x,\delta;u_1(x,\delta),\ldots,u_n(x,\delta)]-\\&-f(c)\,Q_i[x,c;u_1(x,c),\ldots,u_n(x,c)]-\int_c^\delta f'(y)\,Q_i\,\mathrm{d}y=\\&\qquad=\int_c^\delta =f(y)\,F_i\,\mathrm{d}y\end{aligned}\tag{2.4}$$

(If $f(y)$ is discontinuous, the integral $\int_c^d f(y)\,Q_i\,\mathrm{d}y$ must be considered as $\int_c^d Q_i\,\mathrm{d}f(y)$).

3. THE CONCEPT OF THE GENERAL METHOD OF INTEGRAL RELATIONS

Let us choose now a system of groups of the functions $f(y)$.

$$\{[f_{1,1}],[f_{2,1};f_{2,2}],[f_{3,1},f_{3,2}\cdot f_{3,3}],\ldots,[f_{k,1},f_{k,2},\ldots,f_{k,k}],\ldots\}\tag{3.1}$$

that is a system of groups of functions, such that, in the K^{th} group are contained K mutually independent functions (but in the different groups the functions can coincide). We will now divide our rectangular region into K strips and we will set up the functions p_i, Q_i, F_i with the help of a certain interpolated expression involving their values at the boundaries of the strip

$$P_i[x, y; u_1(x, y), \ldots, u_n(x, y)] \cong \sum_{v=0}^{v=K} P_{i,v}^{(x)} \Psi_v(y) \tag{3.2}$$

and similarly for Q_i and F_i.

Here $P_{i,v}(x)$ has the value P_i at the upper limit of the v^{th} strip.

$$P_{iv} = P_i(x, y_v; u_{1,v}, u_{2,v}, \ldots, u_{n,v})$$
$$u_{1,v} = u_1(x, y_v), \ldots.$$

We will then derive the integral relations (2.2) or (2.3) for each of the functions of the K^{th} strip. After substituting in these relations the interpolated expressions (3.2) and completing the integration (either exactly or approximately) the integral relations lead to a system of ordinary differential equations relating to the unknown functions u_i, v. The number of these equations is $n.K$, the number of unknown functions is $n(K+1)$, (since V increases the value from 0 to "K"). The boundary conditions, of which there are probably, in the general case, n at both boundaries $y = c$ and $y = d$, provide the remaining equations. In the case when the upper boundary $y = \delta(x)$ is unknown, one unknown is added, but again probably there is one more additional boundary condition.

Let us look at some individual cases.

If Dirac's δ-functions are used as the functions of the K^{th} group,

$$f_{k,v} = \delta(y - y_v)$$

then we obtain the direct method, in which the derivative $\dfrac{\delta Q_i}{\delta y}$ is obtained by differentiating the interpolated expression (3.2).

By using for the value of $f_{K,v}$ the graded functions

$$f_{K,v} = \left| \begin{array}{ll} 0 & \text{for} \quad y < y_{v-1} \\ 1 & \text{for} \quad y_{v-1} < y < y_v \\ 0 & \text{for} \quad y > y_v \end{array} \right.$$

we obtain the usual method of integral relations. Now we see exactly the meaning of the remark on the "smoothing" of the functions which was made in the introduction. In the direct method, smoothing is achieved with the help of the extremely "discontinuous" δ-functions of Dirac. In the method of integral relations the smoothing process has already occurred with the help of much more continuous functions. It can be expected that the use of still more continuous smoothing functions would enable good results to be obtained even for a small number of strips "K".

In the following we will call the K^{th} approximation the solutions of the system of ordinary differential equations, obtained by dividing the interval into "K" strips.

In the application to boundary layer equations, in place of the functional groups we will use a power of a non-dimensional velocity term thus

$$f_{K,V} = \left(1-\frac{u}{V}\right)^{v}$$

where "u" is the tangential velocity inside the boundary layer, and V the velocity at the outer limit of the layer. Here, it is obvious that it is first necessary to transform the boundary layer equations so that the non-dimensional velocity $\frac{u}{V}$ is one of the independent variables.

We will examine below a boundary layer calculation problem in an incompressible fluid. For a compressible gas, the method remains the same, in principle, but the formulae are much more complicated in form.

4. TRANSFORMATION OF BOUNDARY LAYER EQUATIONS

We will first convert the initial system of boundary layer equations

$$\left.\begin{aligned} & u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = VV'+\nu\frac{\partial^2 u}{\partial y^2} && \text{(a)}\\ & \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0 && \text{(b)}\\ & u=v=0 \quad \text{at} \quad y=0; \quad u=V(x) \quad \text{at} \quad y=\infty && \text{(c)} \end{aligned}\right\} \tag{4.1}$$

to the "standard" boundary conditions and the "standard" type of singularity at the beginning of the boundary layer.

For this we will use new unknown functions

$$\bar{u} = \dot{u}/V, \qquad \bar{v} = v/V\sqrt{\nu} \tag{4.2}$$

and new independent variables

$$\xi = \int_0^x V\,\mathrm{d}x, \qquad \eta = \frac{1}{\sqrt{\nu}}\int_0^y V\,\mathrm{d}y = \frac{Vy}{\sqrt{\nu}} \tag{4.3}$$

Then elementary transformation of the system (4.1) gives the following equation:

$$\left.\begin{aligned} & \bar{u}\frac{\partial \bar{u}}{\partial \xi}+\left(\bar{v}+\frac{\dot{V}}{V}\eta\bar{u}\right)\frac{\partial \bar{u}}{\partial \eta} = \frac{\dot{V}}{V}(1-\bar{u}^2)+\frac{\partial^2 \bar{u}}{\partial \eta^2}\\ & \frac{\partial \bar{u}}{\partial \xi}+\frac{\partial}{\partial \eta}\left(\bar{v}+\frac{\dot{V}}{V}\eta\bar{u}\right) = 0 \end{aligned}\right\} \tag{4.4}$$

$\left(\text{Here } \dot{V} = \frac{\mathrm{d}V}{\mathrm{d}\xi} = \frac{V^1}{V}\right)$

Substituting the symbol

$$w = \bar{v} + \frac{\dot{V}}{V}\eta\bar{u} \tag{4.5}$$

we can write equations (4.4) in a simpler form

$$\left.\begin{aligned} \bar{u}\frac{\partial\bar{u}}{\partial\xi} + w\frac{\partial\bar{u}}{\partial\eta} &= \frac{\dot{V}}{V}(1-\bar{u}^2) + \frac{\partial^2\bar{u}}{\partial\eta^2} \\ \frac{\partial\bar{u}}{\partial\xi} + \frac{\partial w}{\partial\eta} &= 0 \end{aligned}\right\} \tag{4.6}$$

The boundary conditions take the form:

$$\bar{u} = w = 0 \quad \text{at} \quad \eta = 0, \qquad \bar{u} = 1 \quad \text{at} \quad \eta = \infty \tag{4.7}$$

In such a form we see that the boundary conditions are standardized. We will now examine how the solution is obtained at the beginning of the boundary layer.

It is known that when the expansion of V commences with cx^m, then the expansion of the function $\bar{u}$ in a series begins with the term

$$\bar{u} = f_0\left(y/x^{\frac{1-m}{2}}\right)$$

Thence, changing to co-oridinates ξ and η, we obtain

$$\xi \sim \frac{c}{m+1}x^{m+1}, \qquad \eta = \frac{c}{\sqrt{v}}x^m y$$

that is

$$x \sim \left(\frac{m+1}{c}\right)^{\frac{1}{m+1}} \cdot \xi^{\frac{1}{m+1}}$$

$$y \sim \frac{\sqrt{v}}{c}\eta x^{-m} \sim \text{const} \cdot \eta\xi^{-\frac{m}{m+1}}$$

$$y/x^{\frac{1-m}{2}} \sim \text{const} \cdot \frac{\eta}{\sqrt{\xi}}$$

Therefore

$$\bar{u} \sim f_0\left(\text{const} \cdot \frac{\eta}{\sqrt{\xi}}\right) = \psi_0\left(\frac{\eta}{\sqrt{\xi}}\right)$$

for any exponent powers m.

It is found that the singularity at the beginning of the boundary layer is "standardized" in this way.

For all exact solutions of the type $V = cx^m$ the solution of the system (4.6) is always found by the substitution

$$\bar{u} = \varphi'(\eta/\sqrt{2\xi}) \tag{4.8}$$

whence for the functions φ the ordinary differential equation is derived

$$\varphi''' + \varphi\varphi'' + \frac{2m}{m+1}(1-\varphi'^2) = 0 \qquad (4.9)$$

$$\varphi(0) = \varphi'(0) = 0, \quad \varphi'(\infty) = 1$$

We will observe that for exact solutions, the frictional stress at the wall (τ_0) is expressed by the formula

$$\tau_0 = \varrho V^2 \cdot \sqrt{\nu} \left(\frac{\partial \bar{u}}{\partial \eta}\right)_{\eta=0} = \varrho V^2 \sqrt{\nu} \cdot \frac{\varphi''(0)}{\sqrt{2\xi}} \qquad (4.10)$$

In the following section, for simplicity of nomenclature, we will not put a line over u, that is, by "u" we will mean the non-dimensional velocity.

5. INTEGRAL RELATIONS

The following relationship is similar to Kármán's integral condition for the system

$$\frac{d}{d\xi}\int_0^\infty u(1-u)\,d\eta + \frac{\dot{V}}{V}\int_0^\infty (1-u^2)\,d\eta = \left.\frac{\partial u}{\partial \eta}\right|_{\eta=0} \qquad (5.1)$$

which is obtained by subtracting equation (4.6a) from equation (4.6b) multiplied by $(1-u)$ and afterwards integrating the difference obtained with respect to η from 0 to ∞.

We now obtain for the system (4.6) the integral relations of any order, in a similar manner to that which is usually done for a non-transformed system (4.1). For this, we multiply the equation (4.6b) by an arbitrary function $f(u)$, such that it converges sufficiently quickly to zero at $\eta \to \infty$ (for example, $1-u$), and the equation (4.6a) by $f'(u)$ and adding we would then obtain

$$\frac{\partial}{\partial \xi} uf(u) + \frac{\partial}{\partial \eta} wf(u) = \frac{\dot{V}}{V} f'(u)(1-u^2) + f'(u)\frac{\partial^2 u}{\partial \eta^2} \qquad (5.2)$$

Having now integrated this expansion with respect to η from zero to infinity, we obtain

$$\frac{d}{d\xi}\int_0^\infty uf(u)\,d\eta = \frac{\dot{V}}{V}\int_0^\infty f'(u)(1-u^2)\,d\eta - f'(0)\cdot\left.\frac{\partial u}{\partial \eta}\right|_{\eta=0} - \int_0^\infty f''(u)\left(\frac{\partial u}{\partial \eta}\right)^2 d\eta \qquad (5.3)$$

Substitute by Θ the quantity

$$\Theta = \frac{1}{\dfrac{\partial u}{\partial \eta}} \qquad (5.4)$$

Then the integral relation (5.3) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\int_0^1 \Theta u f(u)\,\mathrm{d}u = \frac{\dot{V}}{V}\int_0^1 \Theta f'(u)\,(1-u^2)\,\mathrm{d}u - \frac{f'(0)}{\Theta_0} - \int_0^1 \frac{f''(u)}{\Theta}\,\mathrm{d}u \quad (5.5)$$

In particular, equation (5.1), for which $f(u) = 1-u$ becomes

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\int_0^1 \Theta u(1-u)\,\mathrm{d}u + \frac{\dot{V}}{V}\int_0^1 \Theta(1-u^2)\,\mathrm{d}u = \frac{1}{\Theta_0} \quad (5.6)$$

$\left(\text{Here } \Theta_0 = 1\Big/\left(\frac{\partial u}{\partial \eta}\right)_{\eta=0}\right)$.

6. WORKED EXAMPLE

We now represent the functions Θ and $\frac{1}{\Theta}$ in the K^{th} aproximation by the following expressions

$$\begin{aligned} \Theta &= \frac{1}{1-u}(a_0+a_1u+\cdots+a_{k-1}u^{k-1}) \\ \frac{1}{\Theta} &= (1-u)(b_0+b_1u+\cdots+b_{k-1}u^{k-1}) \end{aligned} \quad (6.1)$$

In place of the system of functions $f_{K,i}$, as has already been said above, we use the power system

$$(1-u)^i$$

The coefficients $a_0\, a_1, \ldots,\; b_0, b_1, \ldots$ are found from the condition that for $u = u_r = \frac{V}{K}$ the functions Θ and $\frac{1}{\Theta}$ would equal their exact values. This leads to the following systems:

I. *The system to a first approximation*

Approximate expressions are:

$$\Theta = \frac{\Theta_0}{1-u}, \quad \frac{1}{\Theta} = \frac{1}{\Theta_0}(1-u) \quad (6.2)$$

Differential equations are:

$$\dot{\Theta} + 3\frac{\dot{V}}{V}\Theta_0 = \frac{2}{\Theta_0} \quad (6.3)$$

(The dot signifies differentiation with respect to 3)

II. *The system to a second approximation*

Approximate expressions are:

$$\Theta=\frac{1}{1-u}[\Theta_0(1-2u)+\Theta_1 u]\,;\quad \frac{1}{\Theta}=(1-u)\left[\frac{1}{\Theta_0}(1-2u)+\frac{1}{\Theta_1}\cdot 4u\right] \qquad (6.4)$$

Differential equations are:

$$\left.\begin{aligned}\dot{\Theta}_0+\frac{\dot{V}}{V}(9\Theta_0+7\Theta_1)&=\frac{34}{\Theta_0}-\frac{32}{\Theta_1}\\ \dot{\Theta}_1+\frac{\dot{V}}{V}(4\Theta_0+6\Theta_1)&=\frac{20}{\Theta_0}-\frac{16}{\Theta_1}\end{aligned}\right\} \qquad (6.5)$$

III. *The system to a third approximation*

Approximate expressions are:

$$\left.\begin{aligned}\Theta&=\frac{1}{1-u}\left[\Theta_0\left(1-\frac{9}{2}u+\frac{9}{2}u^2\right)+\Theta_1 2(2u-3u^2)+\Theta_2\cdot\frac{1}{2}(-u+3u^2)\right]\\ \frac{1}{\Theta}&=(1-u)\left[\frac{1}{\Theta_0}\left(1-\frac{9}{2}u+\frac{9}{2}u^2\right)+\frac{1}{\Theta_1}\cdot\frac{9}{2}(2u-3u^2)-\frac{1}{\Theta_2}\cdot\frac{9}{2}(-u+3u^2)\right]\end{aligned}\right\} \qquad (6.6)$$

Differential equations are:

$$\left.\begin{aligned}\dot{\Theta}_0+\frac{\dot{V}}{V}\left(\frac{67}{2}\Theta_0+40\cdot\Theta_1-\frac{7}{2}\Theta_2\right)&=\frac{225}{\Theta_0}-\frac{234}{\Theta_1}+\frac{9}{\Theta_2}\\ \dot{\Theta}_1+\frac{\dot{V}}{V}\left(\frac{67}{12}\Theta_0+\frac{28}{3}\Theta_1+\frac{13}{12}\Theta_2\right)&=\frac{39}{\Theta_0}-\frac{51}{2}\cdot\frac{1}{\Theta_1}-\frac{12}{\Theta_2}\\ \dot{\Theta}_2+\frac{\dot{V}}{V}\left(-\frac{83}{6}\Theta_0-\frac{52}{3}\Theta_1+\frac{31}{6}\Theta_2\right)&=-\frac{99}{\Theta_0}+\frac{120}{\Theta_1}-\frac{15}{\Theta_2}\end{aligned}\right\} \qquad (6,7)$$

IV. *The system to a fourth approximation*

Approximate expressions are:

$$\left.\begin{aligned}\Theta=\frac{1}{1-u}\Big[&\Theta_0\cdot\frac{1}{3}(3-22u+48u^2-32u^3)+\Theta_1\cdot 3(3u-10u^2+8u^3)+\\ &+\Theta_2(-3u+16u^2-16u^3)+\Theta_3\cdot\frac{1}{3}(u-6u^2+8u^3)\Big]\\ \frac{1}{\Theta}=(1-u)\Big[&\frac{1}{\Theta_0}\cdot\frac{1}{3}(3-22u+48u^2-32u^3)+\\ &+\frac{1}{\Theta_1}\cdot\frac{16}{3}(3u-10u^2+8u^3)+\frac{1}{\Theta_2}\cdot 4(-3u+16u^2-16u^3)+\\ &+\frac{1}{\Theta_3}\cdot\frac{16}{3}(u-6u^2+8u^3)\Big]\end{aligned}\right\} \qquad (6.8)$$

Differential equations are:

$$\left.\begin{aligned}
&\dot{\Theta}_0+\frac{\dot{V}}{V}(97\Theta_0+129\Theta_1-31\Theta_2+3\Theta_3)=\\
&\qquad=940\cdot\frac{1}{\Theta_0}-\frac{3424}{3}\cdot\frac{1}{\Theta_1}+280\cdot\frac{1}{\Theta_2}-96\cdot\frac{1}{\Theta_3}\\
&\dot{\Theta}_1+\frac{\dot{V}}{V}\left(\frac{137}{72}\cdot\Theta_0+\frac{75}{8}\Theta_1+\frac{43}{8}\cdot\Theta_2-\frac{23}{72}\cdot\Theta_3\right)=\\
&\qquad=\frac{133}{6}\cdot\frac{1}{\Theta_0}+\frac{244}{9}\cdot\frac{1}{\Theta_1}-\frac{139}{3}\cdot\frac{1}{\Theta_2}-4\cdot\frac{1}{\Theta_3}\\
&\dot{\Theta}_2+\frac{\dot{V}}{V}\left(-\frac{37}{2}\Theta_0-\frac{45}{2}\cdot\Theta_1+\frac{17}{2}\Theta_2+\frac{1}{2}\Theta_3\right)=\\
&\qquad=-\frac{532}{3}\cdot\frac{1}{\Theta_0}+\frac{688}{3}\cdot\frac{1}{\Theta_1}-52\cdot\frac{1}{\Theta_2}+\frac{16}{3}\cdot\frac{1}{\Theta_3}\\
&\dot{\Theta}_3+\frac{\dot{V}}{V}\left(\frac{177}{8}\Theta_0+\frac{195}{8}\Theta_1-\frac{149}{8}\Theta_2+\frac{41}{8}\Theta_3\right)=\\
&\qquad=\frac{443}{2}\cdot\frac{1}{\Theta_0}-\frac{1060}{3}\cdot\frac{1}{\Theta_1}+175\cdot\frac{1}{\Theta_2}-36\cdot\frac{1}{\Theta_3}.
\end{aligned}\right\} \qquad (6.9)$$

7. EXACT SOLUTIONS. INITIAL CONDITIONS FOR THE APPROXIMATING SYSTEMS OF EQUATIONS

If

$$V=cx^m=c^{\frac{1}{m+1}}(m+1)^{\frac{m}{m+1}}\cdot\xi^{\frac{m}{m+1}} \qquad (7.1)$$

then $\dfrac{\dot{V}}{V}=\dfrac{m}{m+1}\cdot\dfrac{1}{\xi}$, or substituting $\beta=\dfrac{2m}{m+1}$

$$\frac{\dot{V}}{V}=\frac{\beta}{2\xi} \qquad (7.2)$$

In this case the approximating systems also possess an exact solution of the form

$$\Theta_v=A_v\cdot\sqrt{\xi} \qquad (7.3)$$

in which A_v are found as the solutions of the corresponding algebraic systems, as for example, in the first approximation

$$A_0(1+3\beta)=\frac{4}{A_0}$$

in the second approximation

$$(1+9\beta)A_0+7\beta A_1 = \frac{68}{A_0}-\frac{64}{A_1}$$

$$4\beta A_0+(1+6\beta)A_1 = \frac{40}{A_0}-\frac{32}{A_1}$$

and similarly for systems of higher approximations. The quantities A_v are connected with the function φ by an exact differential equation (4.9).

In particular

$$\frac{1}{A_0} = \frac{\varphi''(0)}{\sqrt{2}} \tag{7.4}$$

This permits an appraisal of the accuracy of the approximate systems.

In the table below, such a comparison is set out

β	1st Approx.	2nd Approx.	3rd Approx.	4th Approx.	Exact solution
−0·19	0·32787	separation at	0·14253	0·08514	0·06060
−0·15	0·37081	$\beta \cong -0{\cdot}05$	0·18072	0·14999	0·15299
−0·10	0·51833	. . . .	0·23246	0·22255	0·22576
+0·00	0·50000	0·316492	0·32968	0·33191	0·33206
+0·50	0·79057	0·65628	0·65416	0·65586	0·65597
+1·00	1·00000	0·87247	0·87056	0·87164	0·87157
+1·50	1·17260	1·04538	1·04386	1·04470	1·04456
+2·00	1·32288	1·19371	1·19252	1·19321	1·19304

For an arbitrary velocity law, solutions of the type (7.3) give the initial values for the quantities Θ_v, which permits omission of the singular point $\xi = 0$ and furthermore the application of any methods of numerical integration of the systems of ordinary differential equations.

After determining the quantities Θ_v the boundary layer characteristics c_f, δ^*, δ^{**} are determined in a simple manner

$$c_f = \frac{2\tau_0}{\varrho V^2} = \frac{2\sqrt{\nu}}{\Theta_0}$$

In our equations, Θ and ξ are dimensional quantities. If instead they were non-dimensional functions, taking a certain velocity V_0 for the characteristic velocity and a certain length l for the characteristic length (for example, the velocity in laminar flow and the chord of a streamline profile), then the non-dimensional functions take the following form

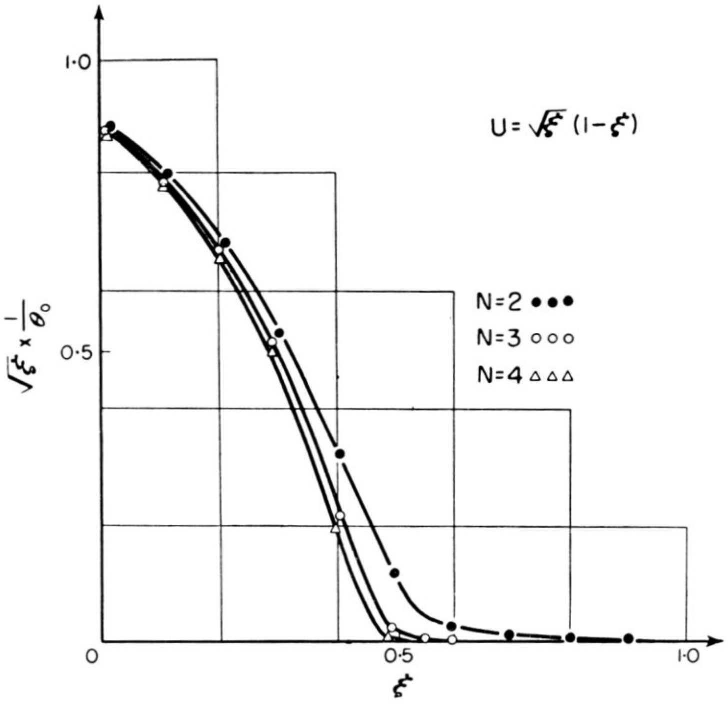
$$\bar{\Theta} = \frac{\Theta}{\sqrt{V_0 l}}, \quad \bar{\xi} = \frac{\xi}{V_0 l}$$

Evidently all the systems of equations (6.3, ..., 6.9) will possess the same form for the variables $\overline{\Theta}$ and $\overline{\xi}$. For non-dimensional quantities, the local frictional coefficient c_f takes the following form

$$c_f = 2\sqrt{\frac{\nu}{V_0 l}} \cdot \frac{1}{\Theta_0} = \frac{2}{\overline{\Theta}_0} \cdot \frac{1}{\sqrt{Re}} \tag{7.5}$$

Expressions for δ^* and δ^{**} will depend on the number of approximations. The general expressions

$$\delta^* = \frac{l}{\sqrt{Re}} \cdot \frac{1}{\overline{V}} \int_0^1 \overline{\Theta}(1-\overline{u})\mathrm{d}\overline{u}; \quad \delta^{**} = \frac{l}{\sqrt{Re}} \cdot \frac{1}{\overline{V}} \int_0^1 \overline{\Theta}\overline{u}(1-\overline{u})\mathrm{d}\overline{u};$$
$$\left(\overline{V} = \frac{V}{V_{0'}}\right) \tag{7.6}$$



are represented clearly, if instead of $\overline{\Theta}$, the approximate expressions (6.2), (6.4), (6.6) or (6.8) are substituted:

$$\frac{\delta^* \overline{V}}{l} \cdot \sqrt{Re} \simeq \Theta_0 \cong \frac{1}{2}\Theta_1 \cong \frac{1}{4}(\Theta_0 + \Theta_2) \cong \frac{1}{2}\Theta_1 - \frac{1}{6}\Theta_2 + \frac{1}{6}\Theta_3$$

$$\frac{\delta^{**}\bar{V}}{l}\cdot\sqrt{Re} \cong \frac{1}{2}\Theta_0 \cong \frac{1}{3}\Theta_1-\frac{1}{6}\Theta_0 \cong \frac{1}{8}\Theta_0-\frac{1}{6}\Theta_1+\frac{5}{24}\Theta_2 \cong$$

$$\cong \frac{1}{90}(-7\Theta_0+27\Theta_1-18\Theta_2+13\Theta_3)$$

for the 1st, 2nd, 3rd and 4th approximations, respectively.

In conclusion I will add an example of the calculation for the case when

$$V = c\sqrt{\xi}\,(1-\xi)$$

or in geometrical co-ordinates

$$V = c\delta h\frac{cx}{2}\bigg/eh^3\frac{cx}{2}$$

The velocity distribution which this formula gives recalls the usual formula for velocity distribution along the profile for large angles of incidence.

On the graph opposite is a comparison of curves for the 2nd, 3rd and 4th approximations. The graph and the table above permit an assessment of the accuracy and rapidity of convergence of the method.